What's Your Identity?

Introduction
Equivalent algebraic expressions, also called algebraic identities, give us a way to express results with numbers that always work a certain way. In this task you will explore several "number tricks" that work because of basic algebra rules. You will extend these observations to algebraic expressions in order to prove polynomial identities. Finally, you will learn and apply methods to expand binomials. It is recommended that you do this task with a partner.

Materials
- Pencil
- Handout
- Calculator

1. First, you will explore an alternate way to multiply two digit numbers that have the same digit in the ten's place.

a. For example, (31)(37) can be thought of as (30 + 1)(30 + 7).

	30	1
30	900	30
7	210	7

Using this model, we find the product equals $900 + 210 + 30 + 7 = 1147$. Verify this solution using a calculator. Are we correct?

Rewrite these similarly and use area models to calculate each of the following products:

b. (52)(57)

c. (16)(13)

d. (48)(42)

e. (72)(75)

2. All of the previous products involved addition, how do you think it would be different if they also included subtraction? What if the products involved both addition and subtraction?

a. (27)(37) can be thought of as (30 - 3)(30 + 7).

	30	-3
30	900	-90
7	210	-21

Using this model, we find the product equals $900 + 210 + -90 + -21 = 999$. Verify this solution using a calculator. Are we correct?

Use both addition and subtraction to rewrite these similarly and use the area models to calculate each of the following products:

b. (46)(57)

c. (16)(25)

d. (38)(42)

e. (62)(75)

3. Look at problem 2d above; is there anything special about the binomials that you wrote and the answer that you got?

a. With a partner compose three other multiplication questions that use the same idea. Explain your thinking. What must always be true for this special situation to work?

Now calculate each of the following using what you have learned about these special binomials.

b. (101)(99)

c. (22)(18)

d. (45)(35)

e. (2.2)(1.8)

4. In Question 3, you computed several products of the form $(x+y)(x-y)$ verifying that the product is always of the form $x^2 - y^2$.

a. If we choose values for x and y so that $x = y$ what will the product be?

b. Is there any other way to choose numbers to substitute for x and y so that the product $(x+y)(x-y)$ will equal 0?

c. In general, if the product of two numbers is zero, what must be true about one of them?

d. These products are called are called conjugates. Give two examples of other conjugates.

e. $(x+y)(x-y)=$ ____________ is called a polynomial **identity** because this statement of equality is true for all values of the variables.

f. Polynomials in the form of $a^2 - b^2$ are called the **difference of two squares**. Factor the following using the identity you wrote in problem 4e:

$x^2 - 25$

$x^2 - 121$

$x^4 - 49$

$4x^4 - 81$

5.. Previously, you've probably been told you couldn't factor the **sum of two squares.** These are polynomials that come in the form $a^2 + b^2$. Well you can factor these; just not with real numbers.

a. Recall $\sqrt{-1} = i$. What happens when you square both sides?

b. Now multiply $(x+5i)(x-5i) =$ ______________________ . Describe what you see.

c. I claim that you can factor the sum of two squares just like the difference of two squares, just with i's after the constant terms. Do you agree? Why or why not?

d. This leads us to another polynomial identity for the sum of two squares.

$a^2 + b^2 =$ ______________________

e. Factor the following using the identity you wrote in problem 5d:

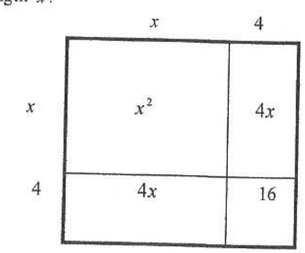
$x^2 + 25$

$x^2 + 121$

$x^2 + 49$

$4x^2 + 81$

6. Now, let's consider another special case to see what happens when the numbers are the same. Start by considering the square below created by adding 4 to the length of each side of a square with side length x.



a. What is the area of the square with side $length = x$?

b. What is the area of the rectangle with $length = x$ and $width = 4$?

c. What is the area of the rectangle with $length = 4$ and $width = x$?

d. What is the area of the square with side $length = 4$?

e. What is the total area of the square in the model above?

f. Draw a figure to illustrate the area of a square with side length $(x + y)$ assuming that x and y are positive numbers. Use your figure to explain the identity for a **perfect square trinomial:** $(x+y)^2 = x^2 + 2xy + y^2$

7. This identity gives a rule for squaring a sum. For example, 103^2 can be written as $(100 + 3)(100 + 3)$. Use this method to calculate each of the following by making convenient choices for x and y.

a. 302^2

b. 54^2

c. 65^2

d. 2.1^2

8. Determine the following identity: $(x - y)^2 =$ ____________________. Explain or show how you came up with your answer.

9. We will now extend the idea of identities to cubes.

a. What is the volume of a cube with side length 4?

b. What is the volume of a cube with side length x?

c. Now we'll determine the volume of a cube with side length $x + 4$.

First, use the rule for squaring a sum to find the area of the base of the cube:

Now use the distributive property to multiply the area of the base by the height, $x + 4$, and simplify your answer:

d. Repeat part 8c for a cube with side length $x + y$. Write your result as a rule for the cube of a sum.

First, use the rule for squaring a sum to find the area of the base of the cube:

Now use the distributive property to multiply the area of the base by the height, $x+y$, and simplify your answer:

e. So the identity for a **binomial cubed** is $(x+y)^3 =$ ____________________

f. Determine the following identity: $(x-y)^3 =$ ____________________.
Explain or show how you came up with your answer.

10. Determine whether the cube of a binomial is equivalent to the sum of two cubes by exploring the following expressions:

a. Simplify $(x+2)^3$

b. Simplify x^3+2^3

c. Is your answer to 10a equivalent to your answer in 10b?

d. Simplify $(x+2)(x^2-2x+4)$

e. Is your answer to part b equivalent to your answer in part d?

f. Your answers to parts b and d should be equivalent. They illustrate two more commonly used polynomial identities:

The Sum of Two Cubes: $a^3+b^3=(a+b)(a^2-ab+b^2)$

The Difference of Two Cubes: $a^3-b^3=(a-b)(a^2+ab+b^2)$

g. Simplify the following and describe your results in words:

$(x-3)(x^2+3x+9)$ $(2x+5)(4x^2-10x+25)$

11. Complete the table of polynomial identities to summarize your findings:

Description	Identity
Difference of Two Squares	$(a+b)(a-b)=$
Sum of Two Squares	$(a+bi)(a-bi)=$
Perfect Square Trinomial	$(a+b)^2=$
Perfect Square Trinomial	$(a-b)^2=$
Binomial Cubed	$(a+b)^3=$
Binomial Cubed	$(a-b)^3=$
Sum of Two Cubes	$a^3+b^3=$
Difference of Two Cubes	$a^3-b^3=$

12. Finally, let's look further into how we could raise a binomial to any power of interest. One way would be to use the binomial as a factor and multiply it by itself n times. However, this process could take a long time to complete. Fortunately, there is a quicker way. We will now explore and apply the binomial theorem, using the numbers in Pascal's triangle, to expand a binomial in $(a+b)^n$ form to the n^{th} power.

Binomial Expansion	Pascal's Triangle	n^{th} row
$(a+b)^0$	1	$n=0$
$(a+b)^1$	1 1	$n=1$
$(a+b)^2$	1 2 1	$n=2$
$(a+b)^3$	1 3 3 1	$n=3$
$(a+b)^4$	1 4 6 4 1	$n=4$

a. Use the fourth row of Pascal's triangle to find the numbers in the fifth row:

Use the fifth row of Pascal's triangle to find the numbers in the sixth row:

Use the sixth row of Pascal's triangle to find the numbers in the seventh row:

b. The binomial coefficients from the third row of Pascal's Triangle are 1, 3, 3, 1, so the expansion of $(x+2)^3 = (1)(x^3)(2^0)+(3)(x^2)(2^1)+(3)(x^1)(2^2)+(1)(x^0)(2^3)$. Describe the pattern you see, and then simplify the result:

c. Use Pascal's triangle in order to expand the following:

$(x+5)^3 =$

$(x+1)^4 =$

$(x+3)^5 =$

d. To expand binomials representing differences, rather than sums, the binomial coefficients will remain the same but the signs will alternate beginning with positive, then negative, then positive, and so on. Simplify the following and compare the result part b.
$(x-2)^3 = (1)(x^3)(2^0)-(3)(x^2)(2^1)+(3)(x^1)(2^2)-(1)(x^0)(2^3)$

e. Use Pascal's triangle in order to expand the following:

$(x-5)^3 =$

$(x-2)^4 =$

$(x-10)^5 =$